Modular symmetries and flavour

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See e.g. https://arxiv.org/pdf/1706.08749 Feruglio

Consider a complex field τ, $(\mathrm{Im}\,\tau > 0)$, a symmetry Γ with elements γ acts

$$\gamma : \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d},$$

It is convenient to represent as 2×2

$$\overline{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} /(\pm\mathbf{1})\,,\ a, b, c, d \in \mathbb{Z},\ \ ad - bc = 1 \right\}.$$

The group has 2 Gens:

$$S_\tau^2 = (S_\tau T_\tau)^3 = \mathbf{1}.$$

$$S_\tau : \tau \to -\frac{1}{\tau}, \qquad T_\tau : \tau \to \tau + 1,$$

$$S_\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A subgroup $\overline{\Gamma}(N)$

$$a = k_a N + 1\,,\ d = k_d N + 1\,,\ b = k_b N\,, \quad c = k_c N\,,$$

$$\overline{\Gamma}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}),\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Make factor group

$$\overline{\Gamma}/\overline{\Gamma}(N) \sim \Gamma_N$$

$\Gamma_2 \sim S_3$, $\Gamma_3 \sim A_4$, $\Gamma_4 \sim S_4$, $\Gamma_5 \sim A_5$

Reminder: Factor Group

G with Inv. subgroup H; Make Cosets

$H, g_i H, \ldots$ These form a group

whose multiplication is

$$g_i H \cdot g_j H = (g_i \cdot g_j) H$$

$$S_3 : \{e, (12), (23), (31); (123), (321)\}$$

$$Z_3 : \{e, (123), (321)\} \equiv H \qquad S_3/Z_3 \sim Z_2$$

$$\{H, (12)H\} \sim Z_2$$

In a Γ_N invariant theory, chiral superfield ϕ

$$\phi_i(\tau) \to \phi_i(\gamma\tau) = (c\tau + d)^{-2k_i} \rho_{I_i}(\gamma)\phi_i(\tau)\,,$$

SUSY action

$$\mathcal{S} = \int d^4x d^2\theta d^2\overline{\theta} K(\phi_i, \overline{\phi}_i; \tau, \overline{\tau}) + \left[\int d^4x d^2\theta W(\phi_i;\tau) + \text{h.c.}\right]\,,$$

$$\begin{aligned} K(\phi_i, \overline{\phi}_i; \tau, \overline{\tau}) &\to K(\phi_i, \overline{\phi}_i; \tau, \overline{\tau}) + f(\phi_i, \tau) + \overline{f}(\overline{\phi}_i, \overline{\tau})\,, \\ W(\phi_i;\tau) &\to W(\phi_i;\tau)\,. \end{aligned}$$ invariant W

$$W(\phi_i;\tau) = \sum_n \sum_{\{i_1,\cdots,i_n\}} \sum_{I_Y} (Y_{I_Y}\phi_{i_1}\cdots\phi_{i_n})_{\mathbf{1}} \cdot$$

"coefficients"

Complicated way of writing a general form

"coefficients" transform

$$Y_{I_Y}(\tau) \to Y_{I_Y}(\gamma\tau) = (c\tau + d)^{2k_Y}\rho_{I_Y}(\gamma)Y_{I_Y}(\tau)$$

Multiples

Multiple modular symmetries as the origin of flavour

Ivo de Medeiros Varzielas, Stephen F. King, Ye-Ling Zhou

We develop a general formalism for multiple moduli and their associated modular symmetries. We apply this formalism to an example based on three moduli with finite modular symmetries S_4^A, S_4^B and S_4^C, associated with two right-handed neutrinos and the charged lepton sector, respectively. The symmetry is broken by two bi-triplet scalars to the diagonal S_4 subgroup. The low energy effective theory involves the three independent moduli fields τ_A, τ_B and τ_C, which preserve the residual modular subgroups Z_3^A, Z_2^B and Z_3^C, in their respective sectors, leading to trimaximal TM_1 lepton mixing, consistent with current data, without flavons.

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Multiple Moduli $\gamma_J : \tau_J \to \gamma_J \tau_J = \frac{a_J \tau_J + b_J}{c_J \tau_J + d_J}\,.$

$\Gamma^1_{N_1} \times \Gamma^2_{N_2} \times \cdots \times \Gamma^M_{N_M},$ Chiral Superfield

$$\begin{aligned}\phi_i(\tau_1, ..., \tau_M) \;&\to\; \phi_i(\gamma_1\tau_1, ..., \gamma_M\tau_M)\\ &= \prod_{J=1,...,M} (c_J\tau_J + d_J)^{-2k_{i,J}} \bigotimes_{J=1,...,M} \rho_{I_{i,J}}(\gamma_J)\phi_i(\tau_1, \tau_2, ..., \tau_M)\,,\end{aligned}$$

SUSY Action:

$$\mathcal{S} = \int d^4x d^2\theta d^2\overline{\theta} K(\phi_i, \overline{\phi}_i; \tau_1, ..., \tau_M, \overline{\tau}_1, ..., \overline{\tau}_M) + \int d^4x d^2\theta W(\phi_i; \tau_1, ..., \tau_M) + \text{h.c.}\,,$$

$$W(\phi_i;\tau_1,\ldots,\tau_M)=\sum_n\sum_{\{i_I,\cdots,i_n\}}\left(Y_{(I_{Y,1},\ldots,I_{Y,M})}\phi_{i_1}\cdots\phi_{i_n}\right)_{\mathbf{1}},$$

where the "coefficients" are modular forms . . .

$$Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1,\ldots,\tau_M)\to Y_{(I_{Y,1},\ldots,I_{Y,M})}(\gamma_1\tau_1,\ldots,\gamma_M\tau_M)$$
$$=\prod_{J=1,\ldots,M}(c_J\tau_J+d_J)^{2k_{Y,J}}\bigotimes_{J=1,\ldots,M}\rho_{I_{Y,J}}(\gamma_J)Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1,\ldots,\tau_M)\,.$$

Complicated way of writing a general form!

Example with S_4

$$S = T_\tau^2 \,,\; T = S_\tau T_\tau \,,\; U = T_\tau S_\tau T_\tau^2 S_\tau \,.$$

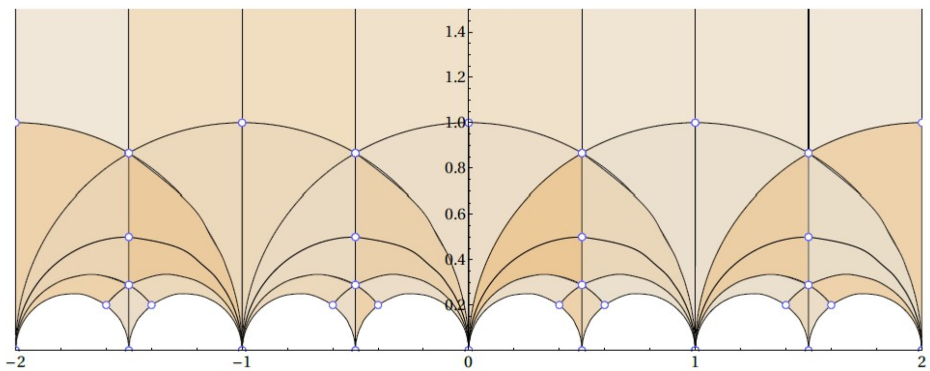
$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \,,\; T = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \,,\; U = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \,.$$

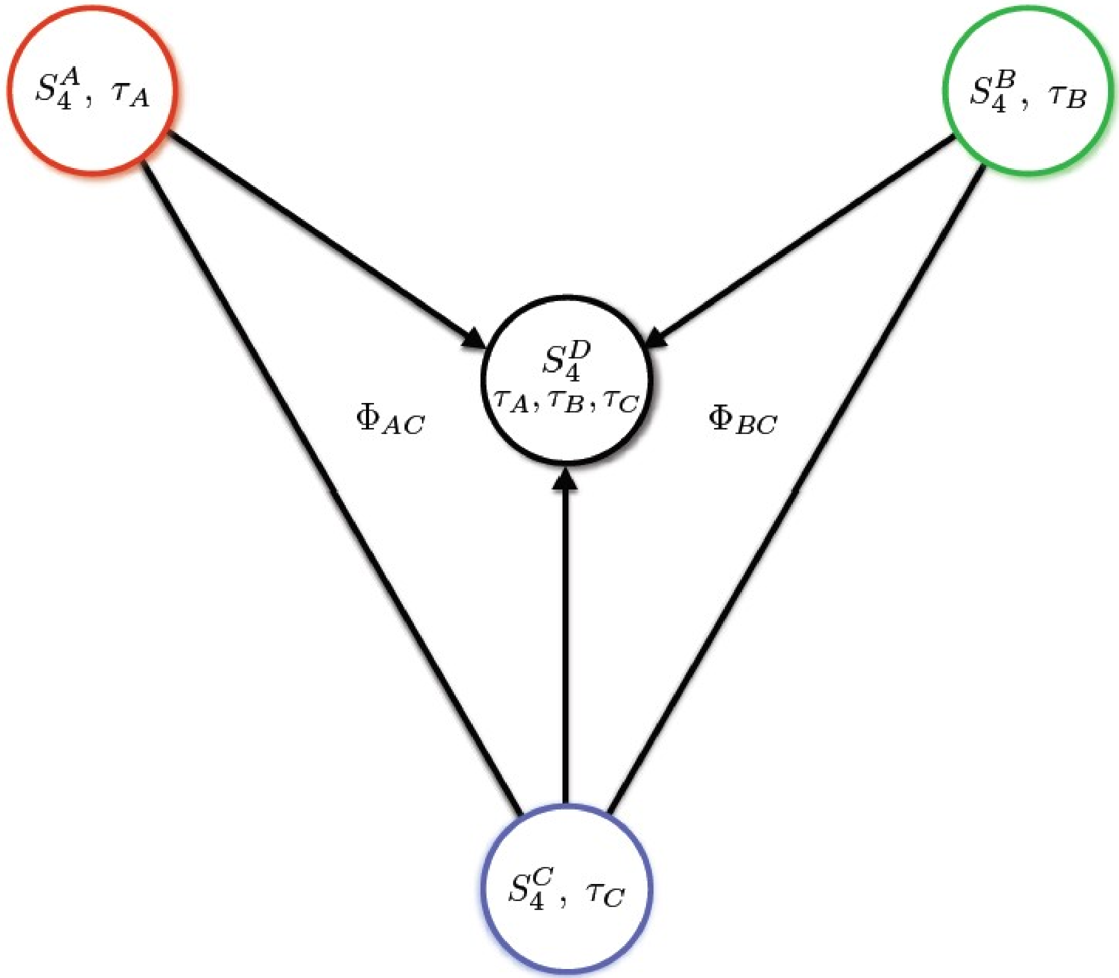
$$\tau_S = i\infty\,,\ \tau_T = \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\,,\ \tau_U = \frac{1}{2} + \frac{i}{2}\,,$$
$$\tau_{TS} = -\omega^2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}\,,\ \tau_{ST} = \frac{1}{2} + \frac{i}{2\sqrt{3}}\,,\ \tau_{STS} = -\frac{1}{2} + \frac{i}{2\sqrt{3}}\,.$$

(S, U & ST were new)

e.g.

$$S : \tau_S \to S\tau_S = \tau_S + 2 = \tau_S\,,$$
$$T : \tau_T \to T\tau_T = \frac{-1}{\tau_T + 1} = \tau_T\,,$$





$S_4^A,\ \tau_A$
$S_4^B,\ \tau_B$
S_4^D
τ_A, τ_B, τ_C
Φ_{AC}
Φ_{BC}
$S_4^C,\ \tau_C$

Field	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
L	**1**	**1**	**3**	0	0	0
e^c	**1**	**1**	**1**	0	0	-6
μ^c	**1**	**1**	**1**	0	0	-4
τ^c	**1**	**1**	**1**	0	0	-2
N_A^c	**1**	**1**	**1**	-6	0	0
N_B^c	**1**	**1**	**1**	0	-4	0
Φ_{AC}	**3**	**1**	**3**	0	0	0
Φ_{BC}	**1**	**3**	**3**	0	0	0

Yuk/Mass	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
$Y_e(\tau_C)$	**1**	**1**	**3**	0	0	6
$Y_\mu(\tau_C)$	**1**	**1**	**3**	0	0	4
$Y_\tau(\tau_C)$	**1**	**1**	**3**	0	0	2
$Y_A(\tau_A)$	**3**	**1**	**1**	6	0	0
$Y_B(\tau_B)$	**1**	**3**	**1**	0	4	0
$M_A(\tau_A)$	**1**	**1**	**1**	12	0	0
$M_B(\tau_B)$	**1**	**1**	**1**	0	8	0
$M_{AB}(\tau_A, \tau_B)$	**1**	**1**	**1**	6	4	0

Φ are bi-Triplets

$$\begin{aligned} w_\ell \;=\; & \frac{1}{\Lambda}\left[L\Phi_{AC}Y_A(\tau_A)N_A^c + L\Phi_{BC}Y_B(\tau_B)N_B^c\right] H_u \\ & + \left[LY_e(\tau_C)e^c + LY_\mu(\tau_C)\mu^c + LY_\tau(\tau_C)\tau^c\right] H_d \\ & + \frac{1}{2}M_A(\tau_A)N_A^cN_A^c + \frac{1}{2}M_B(\tau_B)N_B^cN_B^c + \underbrace{M_{AB}(\tau_A,\tau_B)N_A^cN_B^c}_{\text{cross-term allowed}}\,, \end{aligned}$$

$$\langle \Phi_{AC} \rangle_{i\alpha} = v_{AC} (P_{23})_{i\alpha}\,, \quad \langle \Phi_{BC} \rangle_{m\alpha} = v_{BC} (P_{23})_{m\alpha}\,.$$

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(these come from the Potential)

$$\begin{aligned}
L\Phi_{AC} Y_A(\tau_A) N_A^c &= L_1 \left[(\Phi_{AC})_{11} (Y_A)_1 + (\Phi_{AC})_{21} (Y_A)_3 + (\Phi_{AC})_{31} (Y_A)_2 \right] N_A^c \\
&+ L_2 \left[(\Phi_{AC})_{13} (Y_A)_1 + (\Phi_{AC})_{23} (Y_A)_3 + (\Phi_{AC})_{33} (Y_A)_2 \right] N_A^c \\
&+ L_3 \left[(\Phi_{AC})_{12} (Y_A)_1 + (\Phi_{AC})_{22} (Y_A)_3 + (\Phi_{AC})_{32} (Y_A)_2 \right] N_A^c\,, \\
&= (L_1, L_2, L_3) P_{23} \begin{pmatrix} (\Phi_{AC})_{11} & (\Phi_{AC})_{12} & (\Phi_{AC})_{13} \\ (\Phi_{AC})_{21} & (\Phi_{AC})_{22} & (\Phi_{AC})_{23} \\ (\Phi_{AC})_{31} & (\Phi_{AC})_{32} & (\Phi_{AC})_{33} \end{pmatrix}^T P_{23} \begin{pmatrix} (Y_A)_1 \\ (Y_A)_2 \\ (Y_A)_3 \end{pmatrix} N_A^c\,,
\end{aligned}$$

$$S_4^A \times S_4^B \times S_4^C \to S_4^D \,,$$

$$w_\ell^{\rm eff} \;=\; \left[\frac{v_{AC}}{\Lambda} LY_A(\tau_A)N_A^c + \frac{v_{BC}}{\Lambda} LY_B(\tau_B)N_B^c\right] H_u$$

$$+\left[LY_e(\tau_C)e^c + LY_\mu(\tau_C)\mu^c + LY_\tau(\tau_C)\tau^c\right] H_d$$

$$+\frac{1}{2}M_A(\tau_A)N_A^c N_A^c + \frac{1}{2}M_B(\tau_B)N_B^c N_B^c + M_{AB}(\tau_A,\tau_B)N_A^c N_B^c \,,$$

where e.g. $LY_A(\tau_A)N_A^c \;=\; \left[L_1(Y_A)_1 + L_2(Y_A)_3 + L_3(Y_A)_2\right] N_A^c \,,$

$$Y_e(\langle\tau_C\rangle) = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad Y_\mu(\langle\tau_C\rangle) = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad Y_\tau(\langle\tau_C\rangle) = \begin{pmatrix}0\\1\\0\end{pmatrix},$$

$$Y_A(\langle\tau_A\rangle) = \begin{pmatrix}-1\\2\omega\\2\omega^2\end{pmatrix}, \quad Y_B(\langle\tau_B\rangle) = \begin{pmatrix}0\\1\\-1\end{pmatrix},$$

$$M_N = \begin{pmatrix} M_A & M_{AB} \\ M_{AB} & M_B \end{pmatrix}. \qquad V = e^{i\alpha_3} \begin{pmatrix} \hat{c}_R & \hat{s}_R^* \\ -\hat{s}_R & \hat{c}_R^* \end{pmatrix},$$

cross-term

$$\begin{aligned} M_\nu &= (\mu_1 \hat{c}_R^2 + \mu_2 \hat{s}_R^{*2}) \begin{pmatrix} 1 & -2\omega^2 & -2\omega \\ -2\omega^2 & 4\omega & 4 \\ -2\omega & 4 & 4\omega^2 \end{pmatrix} + (\mu_1 \hat{s}_R^2 + \mu_2 \hat{c}_R^{*2}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \\ &+ (\mu_1 \hat{c}_R \hat{s}_R - \mu_2 \hat{c}_R^* \hat{s}_R^*) \begin{pmatrix} 0 & -1 & 1 \\ -1 & 4\omega^2 & 2i\sqrt{3} \\ 1 & 2i\sqrt{3} & -4\omega \end{pmatrix}, \end{aligned}$$

$\to$ PMNS is TM_1

$$U_{\mathrm{TM}_1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \end{pmatrix}.$$

Good Fit

BF	Para.		χ^2	α_1	α_2	θ_R	μ_1	μ_2
			0.74	64.53°	20.38°	43.01°	$0.00633\,\mathrm{eV}$	$0.0114\,\mathrm{eV}$
	Obs.	θ_{12}	θ_{13}	θ_{23}	δ	m_2	m_3	m_{ee}
		34.33°	8.61°	49.6°	290°	$0.00860\,\mathrm{eV}$	$0.0502\,\mathrm{eV}$	$0.00206\,\mathrm{eV}$

Stabilisers

$$\rho_I(\gamma)Y_I(\tau_\gamma) = (c\tau_\gamma + d)^{-2k}Y_I(\tau_\gamma)\,. \tag{31}$$

This equation lead us to the following important properties for the stabiliser and the modular form:

- A modular form at a stabiliser $Y_I(\tau_\gamma)$ is an eigenvector of the representation matrix $\rho_I(\gamma)$ with respective eigenvalue $(c\tau_\gamma + d)^{-2k}$.
- The stabiliser τ_γ satisfies $|c\tau_\gamma + d| = 1$ since $(c\tau_\gamma + d)^{-2k}$ is an eigenvalue of a unitary matrix.

A special case is that when $(c\tau_\gamma + d)^{-2k} = 1$ is satisfied, $\rho_I(\gamma)Y_I(\tau_\gamma) = Y_I(\tau_\gamma)$, and we recover the residual flavour symmetry generated by γ. In general, the eigenvalue does not need to be fixed at 1 in the framework of modular symmetry.

Stabilisers

Symmetries and stabilisers in modular invariant flavour models

Ivo de Medeiros Varzielas, Miguel Levy, Ye-Ling Zhou

The idea of modular invariance provides a novel explanation of flavour mixing. Within the context of finite modular symmetries Γ_N and for a given element $\gamma \in \Gamma_N$, we present an algorithm for finding stabilisers (specific values for moduli fields τ_γ which remain unchanged under the action associated to γ). We then employ this algorithm to find all stabilisers for each element of finite modular groups for $N = 2$ to 5, namely, $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$. These stabilisers then leave preserved a specific cyclic subgroup of Γ_N. This is of interest to build models of fermionic mixing where each fermionic sector preserves a separate residual symmetry.

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Authors: Ivo de Medeiros Varzielas, Miguel Levy, Ye-Ling Zhou
Title: Symmetries and stabilisers in modular invariant flavour models

Dear Ivo de Medeiros Varzielas,

We are pleased to inform you that your submission JHEP_266P_0820 has been accepted for publication in JHEP.

- $\tau_1 = i$, $\gamma_1 = S_\tau$, $\gamma_2 = e$
- $\tau_2 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$, $\gamma_1 = T_\tau$, $\gamma_2 = S_\tau$
- $\tau_3 = - - \frac{1}{2} + \frac{i\sqrt{3}}{2}$, $\gamma_1 = T_\tau^{N-1}$, $\gamma_2 = S_\tau$
- $\tau_4 = i\infty$, $\gamma_1 = T_\tau$, $\gamma_2 = e$

Method

1. Take $\tau = \tau_i$, where $\tau_i = \gamma_i \tau_i, i = 1, ..., 4$ is a stabiliser of $\mathcal{D}$;

2. Act γ on τ: $\tau' = \gamma\tau$. Compute γ^{-1};

3. The element that stabilises τ' is given by $\gamma^{-1}\gamma_i\,\gamma$.

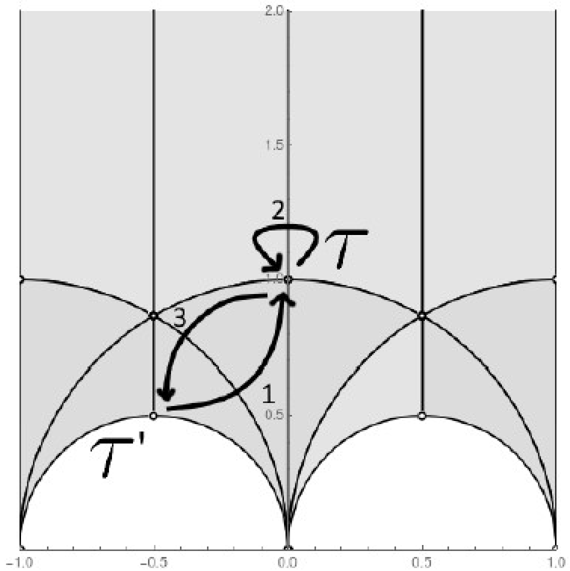
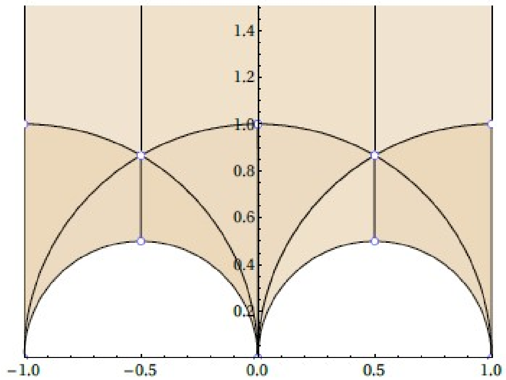


Figure 1: An example of the applied methodology to find the stabilisers of Γ_N. The example shown is for Γ_2, where the arrows denote the actions of different elements, γ^{-1}, γ_i, γ, for 1,2,3 respectively, following the convention of the text.

$\Gamma_2 \simeq S_3$



Domain

Figure 2: The fundamental domain $\mathcal{D}(2)$ of $\bar{\Gamma}(2)$ (i.e., the full target space of $\Gamma_2 \simeq S_3$) with the stabilisers of modular transformations of Γ_2 denoted as dots.

	γ	τ_γ
$\mathcal{C}_2$	$T_\tau C_\tau$	$0, 1+i$
	T_τ	$i\infty, \frac{1}{2}+\frac{i}{2}$
	S_τ	$i, 1$
$\mathcal{C}_3$	$T_\tau S_\tau$	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}, \frac{1}{2}+\frac{i\sqrt{3}}{2}$
	C_τ	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}, \frac{1}{2}+\frac{i\sqrt{3}}{2}$

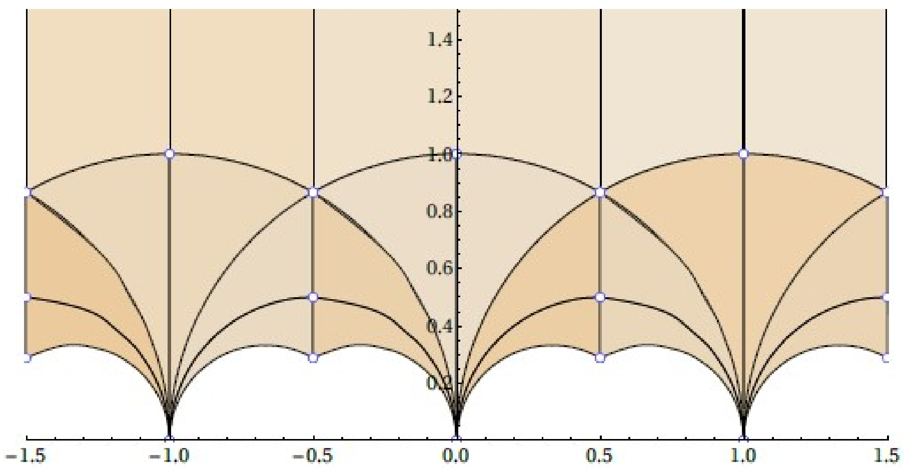
Stabilisers

Table 1: The non-identity elements of Γ_2 and respective stabilisers.

$\Gamma_3 \sim A_4$

Stabilisers

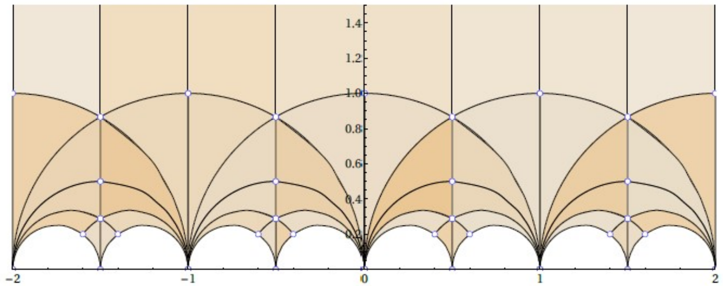
Domain



	γ	τ_γ
$\mathcal{C}_2$	C_τ^2	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}, 1$
	T_τ^2	$i\infty, \frac{3}{2}+\frac{i}{2\sqrt{3}}$
	$T_\tau C_\tau$	$0, \frac{3}{2}+\frac{i\sqrt{3}}{2}$
	$C_\tau T_\tau$	$-1, \frac{1}{2}+\frac{i\sqrt{3}}{2}$
$\mathcal{C}_3$	C_τ	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}, 1$
	T_τ	$i\infty, \frac{3}{2}+\frac{i}{2\sqrt{3}}$
	$C_\tau S_\tau$	$0, \frac{3}{2}+\frac{i\sqrt{3}}{2}$
	$T_\tau S_\tau$	$-1, \frac{1}{2}+\frac{i\sqrt{3}}{2}$
$\mathcal{C}_4$	$T_\tau^2 C_\tau$	$-1+i, \frac{1}{2}+\frac{i}{2}$
	S_τ	$i, \frac{3}{2}+\frac{i}{2}$
	$T_\tau C_\tau T_\tau$	$-\frac{1}{2}+\frac{i}{2}, 1+i$

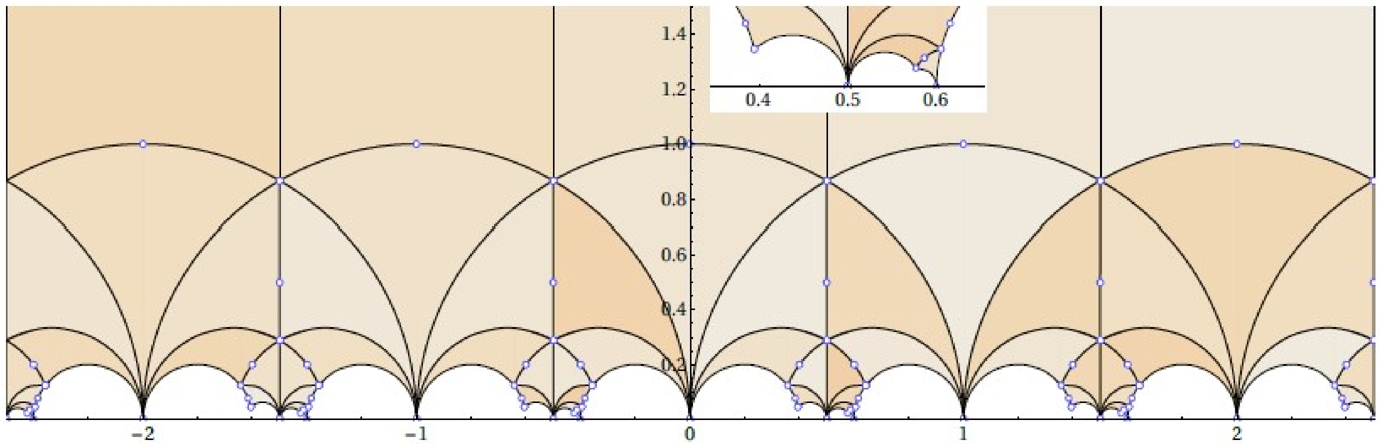
$\Gamma_4 \sim S_4$

	γ	τ_γ
$\mathcal{C}_2$	$T_\tau^2 C_\tau T_\tau$	$\frac{2}{5}+\frac{i}{5}, 2+i$
	$T_\tau^2 C_\tau S_\tau$	$1+i, -\frac{3}{5}+\frac{i}{5},$
	$T_\tau C_\tau T_\tau S_\tau$	$-\frac{3}{2}+\frac{i}{2}, \frac{1}{2}+\frac{i}{2}$
	S_τ	$i, \frac{8}{5}+\frac{i}{5}$
	$C_\tau T_\tau C_\tau$	$-\frac{1}{2}+\frac{i}{2}, \frac{3}{2}+\frac{i}{2}$
	$C_\tau^2 T_\tau$	$-1+i, \frac{7}{5}+\frac{i}{5}$
$\mathcal{C}_3$	C_τ^2	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}, \frac{3}{2}+\frac{i}{2\sqrt{3}}$
	C_τ	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}, \frac{3}{2}+\frac{i}{2\sqrt{3}}$
	$T_\tau^2 C_\tau$	$-\frac{3}{2}+\frac{i\sqrt{3}}{2}, \frac{1}{2}+\frac{i}{2\sqrt{3}}$
	$C_\tau T_\tau C_\tau S_\tau$	$-\frac{3}{2}+\frac{i\sqrt{3}}{2}, \frac{1}{2}+\frac{i}{2\sqrt{3}}$
	$T_\tau C_\tau T_\tau$	$-\frac{1}{2}+\frac{i}{2\sqrt{3}}, \frac{3}{2}+\frac{i\sqrt{3}}{2}$
	$C_\tau^2 T_\tau S_\tau$	$-\frac{1}{2}+\frac{i}{2\sqrt{3}}, \frac{3}{2}+\frac{i\sqrt{3}}{2}$
	$T_\tau C_\tau S_\tau$	$-\frac{3}{2}+\frac{i}{2\sqrt{3}}, \frac{1}{2}+\frac{i\sqrt{3}}{2}$
	$T_\tau S_\tau$	$-\frac{3}{2}+\frac{i}{2\sqrt{3}}, \frac{1}{2}+\frac{i\sqrt{3}}{2}$



$\mathcal{C}_4$	T_τ^2	$i\infty, \frac{3}{2}$
	$C_\tau T_\tau S_\tau$	$0, 2$
	$C_\tau T_\tau C_\tau T_\tau$	$-1, 1$
$\mathcal{C}_5$	T_τ	$i\infty, \frac{3}{2}$
	T_τ^3	$i\infty, \frac{3}{2}$
	$C_\tau S_\tau$	$0, 2$
	$T_\tau C_\tau$	$0, 2$
	$T_\tau^2 S_\tau$	$-1, 1$
	$C_\tau T_\tau$	$-1, 1$

$\Gamma_5 \sim A_5$



	γ	τ_γ
	$T_\tau^3C_\tau T_\tau C_\tau$	$-\frac{3}{2}+\frac{i}{2\sqrt{3}}, \frac{19}{14}+\frac{i\sqrt{3}}{14}$
	$C_\tau T_\tau C_\tau T_\tau^2$	$-\frac{3}{2}+\frac{i}{2\sqrt{3}}, \frac{19}{14}+\frac{i\sqrt{3}}{14}$
	$C_\tau T_\tau C_\tau T_\tau C_\tau$	$-\frac{1}{2}+\frac{i}{2\sqrt{3}}, \frac{33}{14}+\frac{i\sqrt{3}}{14}$
	$C_\tau T_\tau^2 C_\tau$	$-\frac{1}{2}+\frac{i}{2\sqrt{3}}, \frac{33}{14}+\frac{i\sqrt{3}}{14}$
	$T_\tau^3 C_\tau T_\tau^2$	$\frac{15}{26}+\frac{i}{26\sqrt{3}}, \frac{5}{2}+\frac{i\sqrt{3}}{2}$
	$T_\tau^2 C_\tau T_\tau$	$\frac{15}{26}+\frac{i}{26\sqrt{3}}, \frac{5}{2}+\frac{i\sqrt{3}}{2}$
	$T_\tau S_\tau$	$-\frac{37}{26}+\frac{i}{26\sqrt{3}}, \frac{1}{2}+\frac{i\sqrt{3}}{2}$
$\mathcal{C}_2$	$T_\tau C_\tau S_\tau$	$-\frac{37}{26}+\frac{i}{26\sqrt{3}}, \frac{1}{2}+\frac{i\sqrt{3}}{2}$
	$C_\tau T_\tau^2 C_\tau S_\tau$	$-\frac{23}{14}+\frac{i\sqrt{3}}{14}, \frac{1}{2}+\frac{i}{2\sqrt{3}}$
	$T_\tau^2 C_\tau T_\tau C_\tau$	$-\frac{23}{14}+\frac{i\sqrt{3}}{14}, \frac{1}{2}+\frac{i}{2\sqrt{3}}$
	$C_\tau^2 T_\tau^2$	$-\frac{3}{2}+\frac{i\sqrt{3}}{2}, \frac{41}{26}+\frac{i}{26\sqrt{3}}$
	$T_\tau^3 C_\tau$	$-\frac{3}{2}+\frac{i\sqrt{3}}{2}, \frac{41}{26}+\frac{i}{26\sqrt{3}}$
	$C_\tau T_\tau^2 C_\tau T_\tau$	$-\frac{9}{14}+\frac{i\sqrt{3}}{14}, \frac{3}{2}+\frac{i}{2\sqrt{3}}$
	$T_\tau^3 C_\tau T_\tau S_\tau$	$-\frac{9}{14}+\frac{i\sqrt{3}}{14}, \frac{3}{2}+\frac{i}{2\sqrt{3}}$
	$T_\tau^3 C_\tau S_\tau$	$-\frac{41}{26}+\frac{i}{26\sqrt{3}}, \frac{3}{2}+\frac{i\sqrt{3}}{2}$
	$T_\tau C_\tau T_\tau^2$	$-\frac{41}{26}+\frac{i}{26\sqrt{3}}, \frac{3}{2}+\frac{i\sqrt{3}}{2}$
	C_τ	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}, \frac{91}{38}+\frac{i\sqrt{3}}{38}$
	C_τ^2	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}, \frac{91}{38}+\frac{i\sqrt{3}}{38}$
	$T_\tau C_\tau T_\tau^2 C_\tau$	$\frac{5}{14}+\frac{i\sqrt{3}}{14}, \frac{5}{2}+\frac{i}{2\sqrt{3}}$
	$C_\tau T_\tau C_\tau T_\tau S_\tau$	$\frac{5}{14}+\frac{i\sqrt{3}}{14}, \frac{5}{2}+\frac{i}{2\sqrt{3}}$

$\mathcal{C}_3$	$C_\tau^2 T_\tau$	$-1+i, \frac{46}{29}+\frac{i}{29}$
	$T_\tau^2 C_\tau T_\tau^2 C_\tau S_\tau$	$-\frac{8}{13}+\frac{i}{13}, \frac{3}{2}+\frac{i}{2}$
	$C_\tau T_\tau^2 C_\tau T_\tau S_\tau$	$-\frac{12}{5}+\frac{i}{5}, \frac{2}{5}+\frac{i}{5}$
	$T_\tau^2 C_\tau T_\tau^2$	$-\frac{12}{29}+\frac{i}{29}, 2+i$
	S_τ	$-\frac{70}{29}+\frac{i}{29}, i$
	$C_\tau T_\tau^2 C_\tau T_\tau C_\tau$	$-\frac{2}{5}+\frac{i}{5}, \frac{12}{5}+\frac{i}{5}$
	$C_\tau T_\tau C_\tau T_\tau^2 C_\tau$	$-\frac{3}{5}+\frac{i}{5}, \frac{8}{5}+\frac{i}{5}$
	$T_\tau^2 C_\tau T_\tau^2 C_\tau T_\tau$	$\frac{5}{13}+\frac{i}{13}, \frac{5}{2}+\frac{i}{2}$
	$C_\tau^2 T_\tau^2 C_\tau T_\tau$	$-\frac{3}{2}+\frac{i}{2}, \frac{18}{13}+\frac{i}{13}$
	$T_\tau C_\tau T_\tau S_\tau$	$-\frac{21}{13}+\frac{i}{13}, \frac{1}{2}+\frac{i}{2}$
	$C_\tau T_\tau C_\tau T_\tau^2 C_\tau T_\tau$	$-\frac{7}{5}+\frac{i}{5}, \frac{7}{5}+\frac{i}{5}$
	$C_\tau T_\tau C_\tau$	$-\frac{1}{2}+\frac{i}{2}, \frac{31}{13}+\frac{i}{13}$
	$T_\tau^3 C_\tau T_\tau$	$-2+i, \frac{17}{29}+\frac{i}{29}$
	$T_\tau^2 C_\tau S_\tau$	$-\frac{41}{29}+\frac{i}{29}, 1+i$
	$T_\tau C_\tau T_\tau^2 C_\tau T_\tau S_\tau$	$-\frac{8}{5}+\frac{i}{5}, \frac{3}{5}+\frac{i}{5}$

$\mathcal{C}_4$	T_τ	$i\infty, \frac{8}{5}$
	T_τ^4	$i\infty, \frac{8}{5}$
	$T_\tau C_\tau$	$0, \frac{5}{2}$
	$C_\tau S_\tau$	$0, \frac{5}{2}$
	$C_\tau^2 T_\tau S_\tau$	$-\frac{1}{2}, 2$
	$T_\tau C_\tau T_\tau$	$-\frac{1}{2}, 2$
	$T_\tau^3 S_\tau$	$-1, \frac{3}{2}$
	$C_\tau T_\tau$	$-1, \frac{3}{2}$
	$T_\tau^2 S_\tau$	$-\frac{3}{2}, 1$
	$C_\tau T_\tau^2$	$-\frac{3}{2}, 1$
	$C_\tau T_\tau C_\tau S_\tau$	$-2, \frac{1}{2}$
	$T_\tau^2 C_\tau$	$-2, \frac{1}{2}$

$\mathcal{C}_5$	T_τ^2	$i\infty, \frac{8}{5}$
	T_τ^3	$i\infty, \frac{8}{5}$
	$C_\tau T_\tau S_\tau$	$0, \frac{5}{2}$
	$T_\tau C_\tau T_\tau C_\tau$	$0, \frac{5}{2}$
	$C_\tau^2 T_\tau^2 C_\tau T_\tau S_\tau$	$-\frac{1}{2}, 2$
	$T_\tau C_\tau T_\tau^2 C_\tau T_\tau$	$-\frac{1}{2}, 2$
	$C_\tau^2 T_\tau^2 C_\tau$	$-1, \frac{3}{2}$
	$C_\tau T_\tau C_\tau T_\tau$	$-1, \frac{3}{2}$
	$T_\tau^2 C_\tau T_\tau S_\tau$	$-\frac{3}{2}, 1$
	$T_\tau C_\tau T_\tau^2 C_\tau S_\tau$	$-\frac{3}{2}, 1$
	$T_\tau^2 C_\tau T_\tau^2 C_\tau$	$-2, \frac{1}{2}$
	$C_\tau T_\tau C_\tau T_\tau^2 C_\tau S_\tau$	$-2, \frac{1}{2}$

Conclusions

- Modular Symmetries are promising as the origin of Flavour
- Multiple Modular Symmetries have specific advantages
- Stabilisers play a Key role

Thanks

Question about Residual Symmms.

S_4 (S, T, U)

C. Leptons

Z_3 (T)

S, U

$Z_2 \times Z_2$

$\cancel{S}, \cancel{U}$ M_{AB}

Direct Model